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Correlation function of one-dimensional Ising model with power-law potential

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Abstract. A one-dimensional Ising model with a potential falling off like $r^{-(\sigma+1)}$ is considered as a limit of a set of Gaussian models. The spin–spin correlation function is derived, and the critical indices α, β, γ, ν, η are calculated.

1. Introduction

The existence of a phase transition for a one-dimensional Ising model with a power-law potential has been proved by Dyson (1969, 1971), who introduced the hierarchical model for this purpose. Different models, which behave asymptotically as the hierarchical model, have been also considered by Bleher and Sinai (1973, 1975). The critical properties of the hierarchical model with potentral falling off like $r^{-(1+\sigma)}$ have been examined by Kim and Thompson (1977) in the range $0<\sigma<1$, where a phase transition is known to occur, using a renormalisation group recursion relation. Here we introduce a set of Gaussian partition functions from which the Ising partition function is obtained as a limit provided that the constant of interaction is renormalised. We derive an integral equation for the spin–spin correlation function for a wide class of potentials when $h=0$. Critical exponents ν, η, β, γ are then obtained for the power-law potential in agreement with the results of the hierarchical model.

2. Spin–spin correlation function

Consider the one-dimensional Ising model of N spins with the Hamiltonian

$$H=-\tfrac{1}{2}J\sum_{i,j}\rho_{ij}\mu_i\mu_j-h\sum_i\mu_i, \qquad (1)$$

where J is the constant of interaction and $\rho_{ij}=\rho(|j-i|)$. The partition function is

$$Z=\sum_{(\mu_i=\pm1)}\exp(-\beta H),$$

which, using the δ function, can be rewritten in the integral form

$$Z=\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}\prod_{i=1}^{N}[\delta(\mu_i-1)+\delta(\mu_i+1)]\exp(-\beta H)\,\mathrm{d}\mu_1\ldots\mathrm{d}\mu_N. \qquad (2)$$

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Instead of using the δ function, we introduce the Gaussian distributions

$$f_s(\mu_i) = (s/\pi)^{1/2} \exp[-s(\mu_i - 1)^2],$$

and define the set of partition functions

$$Z_s = \left(\frac{s^N}{\pi^N}\right)^{1/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{N} \{\exp[-s(\mu_i - 1)^2] + \exp[-s(\mu_i + 1)^2]\}$$
$$\times \exp(-\beta H)\, d\mu_1 \ldots d\mu_N. \tag{3}$$

The Ising partition function Z can then be obtained by taking the limit of Z_s when $s \to \infty$. Equation (3) can also be written in the form

$$Z_s = \left(\frac{s}{\pi}\right)^{N/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp(-s \sum \mu_i^2 - Ns - \beta H)$$
$$\times \prod_{i=1}^{N} [\exp(2s\mu_i) + \exp(-2s\mu_i)]\, d\mu_1 \ldots d\mu_N$$
$$= \left(\frac{s}{\pi}\right)^{N/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp(-s \sum \mu_i^2 - Ns - \beta H)$$
$$\times \sum_{(v_i = \pm 1)} \exp(-2s \sum v_i\mu_i)\, d\mu_1 \ldots d\mu_N$$
$$= \left(\frac{s}{\pi}\right)^{N/2} \exp(-Ns)$$
$$\times \sum_{(v_i = \pm 1)} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}$$
$$\times \exp[-(\sum a_{ij}\mu_i\mu_j + 2 \sum b_i\mu_i)]\, d\mu_1 \ldots d\mu_N, \tag{4}$$

where

$$a_{ij} = s\delta_{ij} - \tfrac{1}{2}\beta J\rho_{ij}, \qquad b_i = sv_i - \tfrac{1}{2}\beta h.$$

Integral (4) can be immediately evaluated to give

$$Z_s = \left(\frac{s^N}{|\Delta|}\right)^{1/2} \sum_{(v_i = \pm 1)} \exp(\sum a_{ij}^{-1} b_i b_j - sN), \tag{5}$$

where $\Delta = |a_{ij}|$ is the determinant of the matrix a, and $a^{-1} = R(s, \frac{1}{2}\beta J\rho)$ is the resolvent of the matrix $\frac{1}{2}\beta J\rho$.

Differentiating both sides of (5) with respect to $\rho(r)$, where $r = |j - i| = 1, 2, \ldots,$ the derivative of the left-hand side is

$$\partial Z_s/\partial \rho_r = \beta J(N - r)\Gamma(r) Z_s, \tag{6}$$

where $\Gamma(r) = \langle \mu_{j+r}\mu_j \rangle$ is the spin–spin correlation function. The right-hand side contains two terms, $|\Delta|$ and a_{ij}^{-1}, depending on ρ, where

$$\frac{\partial}{\partial \rho_r}\left(\frac{s^N}{|\Delta|}\right)^{1/2} = \frac{1}{2|\Delta|}\left(\frac{s^N}{|\Delta|}\right)^{1/2} \beta J(N - r)A(r).$$

$A(r)$, the co-factor of a_{ij}, can be calculated from the relation

$$A(r) = \Delta a_{ij}^{-1}.$$

Using the Fourier transforms

$$a(r)=\frac{1}{2\pi}\int_0^{2\pi}\tilde{a}(q)\exp(-\mathrm{i}qr)\,\mathrm{d}q,\qquad \tilde{a}(q)=\sum_{-\infty}^{\infty}a(r)\exp(\mathrm{i}qr),$$

and noting that in the q representation the matrix a is diagonal, that is

$$\tilde{a}^{-1}(q)=1/\tilde{a}(q)=1/s-\tfrac{1}{2}\beta J\tilde{\rho}(q),$$

we get

$$a^{-1}(r)=\frac{1}{2\pi}\int_0^{2\pi}\frac{\exp(-\mathrm{i}qr)}{s-\frac{1}{2}\beta J\tilde{\rho}(q)}\,\mathrm{d}q.$$

Therefore

$$\frac{\partial}{\partial\rho_r}\left(\frac{s^N}{|\Delta|}\right)^{1/2}=\frac{1}{2}\left(\frac{s^N}{|\Delta|}\right)^{1/2}\beta J(N-r)\frac{1}{2\pi}\int_0^{2\pi}\frac{\exp(-\mathrm{i}qr)}{s-\frac{1}{2}\beta J\tilde{\rho}(q)}\,\mathrm{d}q. \tag{7}$$

Similarly

$$\frac{\partial}{\partial\rho_r}\exp\Big(\sum_{i,j}a_{ij}^{-1}b_ib_j\Big)=\exp\Big(\sum_{i,j}a_{ij}^{-1}b_ib_j\Big)\sum_{l,m}b_lb_m\frac{1}{2a}\int_0^{2\pi}\frac{-\frac{1}{2}\beta J\exp[-\mathrm{i}q(r-l-m)]}{[s-\frac{1}{2}\beta J\tilde{\rho}(q)]^2}\,\mathrm{d}q. \tag{8}$$

If $\frac{1}{2}\|\rho\|\beta J<s$, where $\|\rho\|<(\sum_{i,j}\rho_{ij}^2)^{1/2}<\infty$, then the resolvent $R(s,\frac{1}{2}\beta J\rho)$ can be expanded in a power series, so that

$$R(s,\tfrac{1}{2}\beta J\rho)=\sum_0^{\infty}\frac{(\frac{1}{2}\beta J\rho)^n}{s^{n+1}}.$$

Therefore

$$\begin{aligned}-\beta H'(h)&=\sum_{i,j}a_{ij}^{-1}b_ib_j\\ &=s\sum_i v_i^2+\tfrac{1}{2}\beta J\sum_{i,j}\rho_{ij}v_iv_j-\beta h\sum_i v_i+\mathrm{O}(J/s)\\ &\simeq s\sum_i v_i^2-\beta H(-h).\end{aligned}$$

If the magnetic field $h=0$, the derivative of (4) finally takes the form

$$\Gamma(r)=\left(\frac{s^N}{|\Delta|}\right)^{1/2}\frac{1}{2\pi}\int_0^{\pi}\frac{\cos(qr)}{s-\frac{1}{2}\beta J\tilde{\rho}(q)}\,\mathrm{d}q-\frac{1}{\pi}\left(\frac{s^N}{|\Delta|}\right)^{1/2}\int_0^{\pi}\frac{\Gamma(q)\cos(qr)}{[1-(\beta J/2s)\tilde{\rho}(q)]^2}\,\mathrm{d}q. \tag{9}$$

This integral equation defines the spin–spin correlation function for all potentials satisfying the conditions: $\rho_{ij}=\rho(|j-i|)$, $\|\rho\|$ finite, and the temperature $T>\frac{1}{2}\|\rho\|J/s$.

3. Spin–spin correlation function for a power law potential

If the potential falls off by a power law, that is $\rho_{ij}\sim|j-i|^{-\sigma-1}$ when $|j-i|$ is large, the Fourier transform of $\rho(r)$ is given by

$$\tilde{\rho}(q)=b-a|q|^{\sigma}, \tag{10}$$

where $\sigma>-1$, $a=\pi/\Gamma(\sigma+1)\sin(\pi\sigma/2)$ and $b=(2\pi)^{\sigma}a/(\sigma+1)$.

The behaviour of the integral on the left-hand side of (9), which after substitution from (10) is

$$I=\int_0^{\pi}\frac{\cos(qr)}{s-\frac{1}{2}\beta Jb+\frac{1}{2}\beta Ja|q|^{\sigma}}\,\mathrm{d}q \qquad \text{(at large } r\text{)}, \tag{11}$$

can be obtained by making the transformation $y^2=(qr)^{\sigma}$ and considering the contour integral

$$\int_{\mathrm{C}}\frac{\exp(\mathrm{i}Z^{2/\sigma})Z^{(2/\sigma)-1}}{s-\frac{1}{2}b\beta J+\frac{1}{2}a\beta JZ^{2}/r^{\sigma}}\,\mathrm{d}Z.$$

The contour C consists of the upper half of the circle $|Z|=r\pi\,(r\to\infty)$ bounded by the real axis and a small circle round the origin. The asymptotic behaviour for large r is determined by the residue of the pole

$$Z=\left(\frac{s-\frac{1}{2}b\beta J}{\frac{1}{2}a\beta J}\right)^{1/2}r^{\sigma/2}\mathrm{i} \qquad (\mathrm{i}=\sqrt{-1}).$$

The asymptotic behaviour of the spin–spin correlation function for large $|j-i|$ and s is therefore given by

$$\Gamma(|j-i|)\sim\frac{2\pi}{\sigma\beta Ja}\left(\frac{s-\frac{1}{2}b\beta J}{\frac{1}{2}a\beta J}\right)^{(1/\sigma)-1}\exp\left[-\left(\frac{s-\frac{1}{2}b\beta J}{\frac{1}{2}a\beta J}\right)^{1/\sigma}|j-i|\sin\left(\frac{\pi}{\sigma}\right)\right]$$
$$\times\sin\left[\left(\frac{s-\frac{1}{2}b\beta J}{\frac{1}{2}a\beta J}\right)^{1/\sigma}|j-i|\cos\left(\frac{\pi}{\sigma}\right)+\sin\left(\frac{\pi}{\sigma}\right)\right]. \tag{12}$$

Hence the critical temperature is

$$T_{\mathrm{c}}=bJ/2s. \tag{13}$$

4. Critical exponents

From (12), the spin–spin correlation function in the neighbourhood of the critical temperature at large distance $|j-i|$ is determined by

$$\Gamma(j-i)\sim\frac{2\pi}{\sigma\beta Ja}\left(\frac{2s}{a\beta_{\mathrm{c}}J}\right)^{(1/\sigma)-1}\sin\left(\frac{\pi}{\sigma}\right)\left(\frac{T-T_{\mathrm{c}}}{T_{\mathrm{c}}}\right)^{(1/\sigma)-1}$$
$$\times\exp\left[-\left(\frac{T-T_{\mathrm{c}}}{T_{\mathrm{c}}}\right)^{1/\sigma}\left(\frac{2s}{a\beta_{\mathrm{c}}J}\right)^{1/\sigma}|j-i|\sin\left(\frac{\pi}{\sigma}\right)\right]\exp\left(-\frac{|j-i|}{\xi}\right). \tag{14}$$

The correlation length is

$$\xi\sim(T-T_{\mathrm{c}})^{-1/\sigma};$$

hence the exponent

$$\nu=1/\sigma. \tag{15}$$

At the critical temperature, the integral (11) can be directly evaluated to give

$$I=\frac{2}{\beta Ja}\int_0^{\pi}q^{-\sigma}\cos(q|j-i|)\,\mathrm{d}q=\frac{2T_{\mathrm{c}}}{Ja}\frac{\Gamma(-\sigma+1)\cos[\frac{1}{2}\pi(1-\sigma)]}{|j-i|^{-\sigma+1}}.$$

Since

$$\Gamma(j-i) \sim 1/|j-i|^{\eta-1}$$

at $T = T_c$, the exponent

$$\eta = 2 - \sigma. \tag{16}$$

The susceptibility is determined from

$$\chi = \frac{1}{T}\sum_r \Gamma(r) \sim \frac{(T-T_c)^{(1/\sigma)-1}}{1-\exp\left[-\left(\frac{T-T_c}{T_c}\right)^{1/\sigma}\left(\frac{2s}{a\beta_c J}\right)^{1/\sigma}\sin\left(\frac{\pi}{\sigma}\right)\right]} \sim (T-T_c)^{-1};$$

hence the exponent $\gamma = 1$.

Assuming that equation (9) is valid when $T < T_c$, the asymptotic behaviour is then determined by the two real poles

$$Z = \pm\left(\frac{\frac{1}{2}\beta J b - s}{\frac{1}{2}a\beta J}\right)^{1/2} r^{\sigma/2}$$

which are surrounded by two small semicircles in the upper half Z plane, giving

$$\Gamma(j-i) \sim \frac{\pi}{\beta_c J a \sigma}\left(\frac{T_c - T}{T_c}\right)^{(1/\sigma)-1} \cos\left(\frac{2\pi}{\sigma}\right).$$

As $\Gamma(j-i)$ tends to the square of the spontaneous magnetisation when $|j-i| \to \infty$, then

$$M^2 \sim (T_c - T)^{(1/\sigma)-1};$$

therefore the exponent

$$\beta = \tfrac{1}{2}[(1/\sigma) - 1]. \tag{17}$$

Also, from $\xi \sim (T-T_c)^{-2+\alpha}$, we get $\alpha = 2 - 1/\sigma$, so the exponents α, β, γ satisfy the scaling relation $\alpha + 2\beta + \gamma = 2$.

5. Ising model

Finally we notice that the Ising model ($s \to \infty$) exhibits a phase transition if the constant of interaction $J \to \infty$; therefore the constant of interaction must be renormalised such that the ratio J/s is finite. The critical temperature in this case is $T_c = \frac{1}{2}bJ_{ph}$, where J_{ph} is the renormalised constant. This renormalisation procedure corresponds to a renormalisation of the magnetic moment μ_i in the Hamiltonian such that

$$\mu_i = (1/\sqrt{s})\mu_{ph},$$

and hence

$$\Gamma(r) = (1/s)\Gamma_{ph}(r).$$

From (11) the renormalised spin–spin correlation function near T_c is

$$\Gamma_{ph}(|j-i|) \sim \frac{2\pi}{\sigma\beta_c J_{ph}}\left(\frac{2}{a\beta_c J_{ph}}\right)^{(1/\sigma)-1} \sin\left(\frac{\pi}{\sigma}\right)\left(\frac{T-T_c}{T_c}\right)^{(1/\sigma)-1}$$
$$\times \exp\left[-\left(\frac{T-T_c}{T_c}\right)^{1/\sigma}\left(\frac{2}{a\beta_c J_{ph}}\right)^{1/\sigma} |j-i| \sin\left(\frac{\pi}{\sigma}\right)\right]. \tag{18}$$

6. Conclusions and discussion

In this paper we have introduced a model suitable for the study of the critical behaviour of the Ising model with long-range interactions. The spin–spin correlation function is derived in (9) for a large class of potentials. This formula is then applied analytically in § 4 to calculate the critical exponents for potentials falling off like $r^{-\sigma-1}$. Integral (11) is solved exactly at $T = T_c$, from which we get the exponent $\eta = 2-\sigma$ for all $0<\sigma\leq 1$. The same result has been also obtained by Kim and Thompson (1977) in the same range.

However, the exponents ν, γ and β are obtained from the asymptotic behaviour of the spin–spin correlation function as $\gamma\to\infty$. The asymptotic behaviour of (11) is determined by the pole of the corresponding complex integral in the upper half Z plane. In fact the results obtained for ν, γ and β are correct if the contribution of the large semicircle at infinity is less than the pole contribution.

The study of the behaviour of this integral shows that the contribution of the large semicircle vanishes if $1/\sigma = n+p$, where $n = 1, 2, 3, \ldots$, and $0\leq p<\frac{1}{2}$, i.e. $2/(1+2n)<\sigma\leq 1/n$, and diverges as $r\to\infty$ if $\frac{1}{2}<\sigma\leq\frac{2}{3}$. Our results for ν, γ and β are therefore true in the intervals $2/(1+2n)<\sigma\leq 1/n$, where $n = 1, 2, \ldots$. The values obtained for ν, γ and β are in agreement with those of previous workers in the range $0<\sigma<\frac{1}{2}$. Different behaviour of the exponents is to be expected in the region $\frac{1}{2}<\sigma\leq\frac{2}{3}$. In this region the resolvent $R(s, \frac{1}{2}\beta J\rho)$ cannot be expanded in the form

$$R(s, \tfrac{1}{2}\beta J\rho) = \sum_{0}^{\infty} \frac{(\frac{1}{2}\beta J\rho)^n}{s^{n-1}}. \tag{19}$$

In fact equation (9) is useful for the calculation of the spin–spin correlation function of the Gaussian model introduced if $J/s<2T/\|\rho\|$, which is the condition of convergence of the series (19).

For the Ising model $s\to\infty$ and therefore J must tend to infinity in order to have non-vanishing critical temperature and spin–spin correlation function. If the renormalised constant $J_{\mathrm{ph}}<2T/\|\rho\|$, then the values of the exponents ν, γ and β derived in § 4 are the approximate physical values, as only the first and second terms in the expansion of the resolvent are taken into consideration in the derivation of (9). This explains why we obtain the same values for the exponents in the region $\frac{2}{3}<\sigma\leq 1$ as in the classical region $0<\sigma\leq\frac{1}{2}$. Better results can be obtained if more terms in the resolvent expansion are included. Our results in this approximation agree with the hierarchical model (HM) and the long-range Ising model in the range $0<\sigma\leq\frac{1}{2}$. In the range $\frac{2}{3}<\sigma\leq 1$ the behaviour of the exponents disagrees with the HM and is only in rough agreement with the Ising model. This is because the critical temperature in the HM approaches zero as $\sigma\to 1$, and at $\sigma = 1$ the HM exhibits singular behaviour, while the Ising model calculations show that a phase transition exists in this region including $\sigma = 1$.

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